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A Kind of Four-Mode Continuous-Variable Entangled State Generated by Beamsplitter and Parametric Down-Conversions

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Abstract We introduce a new four-mode continuous-variable entangled state in Fock space which can compose a new quantum mechanical representation. This state is important and can be common used because it can be generated by using one symmetrical beamsplitter and a pair of parameter down-convertors. Applying this new state in quantum information is described.

Keywords Four-mode entangled states · IWOP technique · Application in quantum teleportation

1 Introduction

The conception of quantum entanglement has become more and more fascinating and important, since the publication of Einstein–Podolsky–Rosen' (EPR) paper in 1935 [1], arguing the incompleteness of quantum mechanics. Quantum entanglement now plays a central role in quantum information processing [2–4]. Recently quantum information and quantum computation have been extended to the domain of quantum states of continuous variables (CV) [5, 6], due to the relative simplicity and high efficiency in generation, manipulation and detection of the CV state. The investigation of CV multipartite entangled states has made significant progress in theory and experiment. Multipartite quantum protocols were proposed and performed experimentally, such as quantum teleportation networks [7], controlled dense coding [8], quantum secret sharing [9] and quantum telecloning (quantum nonlocal cloning) [10] based on tripartite entanglement. In Ref. [10], to demonstrate $1 \to 2$ quantum telecloning of coherent states, Koike et al created a tripartite entanglement. The scheme for creating the tripartite entanglement is as follows: Two individual squeezed vacuum modes are combined at a 50:50 beamsplitter (BS) with a $\pi/2$ phase shift and then one output mode is divided into two modes with another 50:50 BS. However, there is no any explicit form of entangled state constructed in Ref. [10].

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For EPR's entanglement, Fan et al [11, 12] constructed a two-mode entangled state and a parametrized entangled state in Fock space, later Hu and Fan [13–15] proposed the three-mode CV entangled state which can be prepared by using a asymmetric BS [16] and a parametric down-conversion (PDC) instrument [17–22]. Both two kinds of entangled states are proved to make up new quantum mechanical representations. Multipartite entanglement in pure states of many systems is a founding property and a crucial resource for quantum information science, its complete theoretical understanding can be deepen by virtue of CV representation theory.

In this paper, we shall introduce a new four-mode continuous-variable entangled state in Fock space, denoted by $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$, which can compose a new quantum mechanical representation. This state is important and can be commonly used because it can be generated by using one symmetrical beamsplitter (BS) and a pair of parameter down-convertors (PDCs). The work is arranged as follows: In Sect. 2 we construct and analyze the new entangled state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$. In Sect. 3 we design the experimental setup to implement it. Sect. 4 is devoted to deriving its completeness property and partly non-orthogonal property. The Schmidt decomposition and quadrature amplitude measurement of the new state is demonstrated in Sect. 5. An application of the $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ in quantum teleportation is considered in the last section.

2 The New Four-Mode Entangled State

We now write down $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ in Fock space:

$$\begin{aligned}|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = \operatorname{sech}\lambda\operatorname{sech}\mu\exp\Big\{&-\frac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2)\\&+(a_1^\dagger-\alpha^*)(a_2^\dagger-\beta^*)\tanh\lambda+a_3^\dagger(a_4^\dagger-\gamma^*)\tanh\mu\\&+(a_1^\dagger-\alpha^*)a_3^\dagger\operatorname{sech}\lambda\operatorname{sech}\mu+\alpha a_1^\dagger+\beta a_2^\dagger+\gamma a_4^\dagger\Big\}|0000\rangle,\end{aligned} \tag{1}$$

where $|0000\rangle$ is annihilated by a_i, $i=1,2,3,4$, λ and μ are two real numbers (i.e., two squeezing parameters characterizing two PDCs), see Fig. 1. In particular, when $\lambda=\mu=0$,

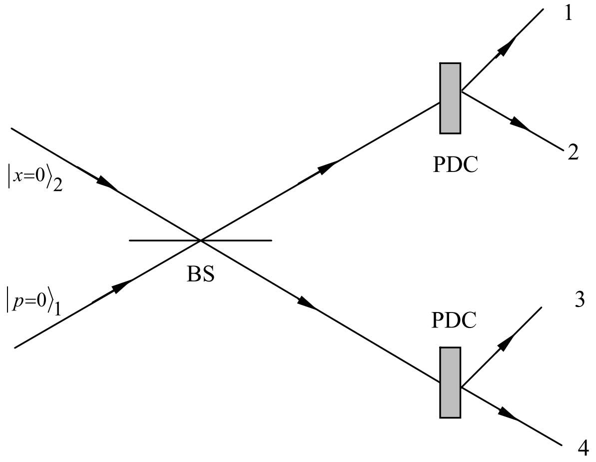

Fig. 1 The schematic diagram of creating the four-mode entangled state. BS and PDC denote the beamsplitter and parameter down-conversion, respectively

(1) reduces to

$$|\alpha,\beta,\gamma\rangle_{0,0} = \exp\left[-\frac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2+(a_1^\dagger-\alpha^*)a_3^\dagger + \alpha a_1^\dagger+\beta a_2^\dagger+\gamma a_4^\dagger)\right]|0000\rangle$$
$$= |\eta=\alpha\rangle_{1,3}\otimes|\beta\rangle_2\otimes|\gamma\rangle_4, \tag{2}$$

where $|\eta=\alpha\rangle_{13}$ is the bipartite entangled state in $a_1^\dagger$-$a_3^\dagger$ mode [13–15], while $|\beta\rangle_2$ and $|\gamma\rangle_4$ are the coherent states in $a_2^\dagger$ and $a_4^\dagger$ modes, respectively. In addition, (1) will reduce to a tripartite entangled state with $\gamma=\mu=0$. Operating a_i on $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ we obtain four independent eigenvector equations

$$(a_4 - a_3^\dagger\tanh\mu)|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = \gamma|\alpha,\beta,\gamma\rangle_{\lambda,\mu}, \tag{3}$$

$$(a_2 - a_1^\dagger\tanh\lambda)|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = (\beta-\alpha^*\tanh\lambda)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}, \tag{4}$$

$$(a_1 - a_2^\dagger\tanh\lambda - a_3^\dagger\operatorname{sech}\lambda\operatorname{sech}\mu)|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = (\alpha-\beta^*\tanh\lambda)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}, \tag{5}$$

$$(a_3 - a_1^\dagger\operatorname{sech}\lambda\operatorname{sech}\mu - a_4^\dagger\tanh\mu)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$$
$$= -(\gamma^*\tanh\mu+\alpha^*\operatorname{sech}\lambda\operatorname{sech}\mu)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}. \tag{6}$$

By introducing $X_i = (a_i+a_i^\dagger)/\sqrt{2}$, $P_i=(a_i-a_i^\dagger)/(\sqrt{2}\mathrm{i})$ and combining (5) and (6) we have

$$[(X_1\cosh\lambda - X_2\sinh\lambda)\operatorname{sech}\mu - X_3 + X_4\tanh\mu]|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$$
$$= \sqrt{2}[(\alpha_1\cosh\lambda-\beta_1\sinh\lambda)\operatorname{sech}\mu+\gamma_1\tanh\mu]|\alpha,\beta,\gamma\rangle_{\lambda,\mu}, \tag{7}$$

$$[(P_1\cosh\lambda + P_2\sinh\lambda)\operatorname{sech}\mu + P_3 + P_4\tanh\mu]|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$$
$$= \sqrt{2}[(\alpha_2\cosh\lambda+\beta_2\sinh\lambda)\operatorname{sech}\mu+\gamma_2\tanh\mu]|\alpha,\beta,\gamma\rangle_{\lambda,\mu}. \tag{8}$$

Note that $|\alpha,\beta,\gamma\rangle_{\lambda\mu}$ is the common eigenvector of the above four operators appearing on the left-hand side of (3,4–7,8), which constitute a complete commutable operator set.

3 The Setup for Generating $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$

We now discuss an experimental setup to produce the ideal four-mode entangled state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$. Recall that the simultaneous eigenstate of two particles' relative coordinate X_1-X_2 and total momentum P_1+P_2 in two-mode Fock space was constructed as [11, 12]

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2+\eta b_1^\dagger-\eta^* b_2^\dagger + b_1^\dagger b_2^\dagger\right]|00\rangle_{12}, \quad \eta=(\eta_1+\mathrm{i}\eta_2)/\sqrt{2}. \tag{9}$$

By introducing the displacement operator $D_1(\eta)=\exp(\eta b_1^\dagger - \eta^* b_1)$, $|\eta\rangle$ can be expressed as

$$|\eta\rangle = D_1(\eta)\exp[b_1^\dagger b_2^\dagger]|00\rangle_{12}. \tag{10}$$

In particular, $|\eta=0\rangle=\exp[b_1^\dagger b_2^\dagger]|00\rangle_{12}$, which can be expressed as

$$\exp[b_1^\dagger b_2^\dagger]|00\rangle_{12}=B_{12}\left(\frac{\pi}{4}\right)|p=0\rangle_1\otimes|x=0\rangle_2, \tag{11}$$

where $|p=0\rangle_1\sim\exp(\frac{1}{2}a_1^{\dagger 2})|0\rangle_1$ is the zero-momentum eigenstate and $|x=0\rangle_2\sim\exp(-\frac{1}{2}a_2^{\dagger 2})|0\rangle_2$ is the zero-position eigenstate (they can be considered as single-mode maximally squeezed state in p and x directions, respectively). Equation (11) shows that the state $|\eta=0\rangle$ can be produced when a pair of incident squeezed rays fall on a symmetric 50:50 BS (without loss and phase shift), the operation of BS is represented by the operator $B_{12}(\frac{\pi}{4})=\exp[\frac{\pi}{4}(b_1^\dagger b_2-b_1 b_2^\dagger)]$ [23–25].

In similar to the formation shown in (11), based on the state $\exp[a_1^\dagger a_3^\dagger]|00\rangle_{13}$ generated by a beamsplitter, we further make two squeezing transformation for $a_1^\dagger$ and $a_3^\dagger$ through two parametric down-convertors [16]

$$S_{12}a_1^\dagger S_{12}^{-1}=a_1^\dagger\cosh\lambda-a_2\sinh\lambda, \tag{12}$$

$$S_{34}a_3^\dagger S_{34}^{-1}=a_3^\dagger\cosh\mu-a_4\sinh\mu, \tag{13}$$

by the two-mode squeezing operator $S_{ij}=\exp[\lambda(a_i^\dagger a_j^\dagger-a_i a_j)]$, then the output state is

$$\begin{aligned}
&S_{12}S_{34}\exp[a_1^\dagger a_3^\dagger]|0000\rangle\\
&\quad=\exp[(a_1^\dagger\cosh\lambda-a_2\sinh\lambda)(a_3^\dagger\cosh\mu-a_4\sinh\mu)]S_{12}S_{34}|0000\rangle\\
&\quad=\operatorname{sech}\lambda\operatorname{sech}\mu\exp[a_1^\dagger a_3^\dagger\cosh\lambda\cosh\mu]\exp[-a_1^\dagger a_4\cosh\lambda\sinh\mu]\\
&\qquad\times\exp[-a_2 a_3^\dagger\cosh\mu\sinh\lambda]\exp[a_2 a_4\sinh\lambda\sinh\mu]\\
&\qquad\times\exp[a_1^\dagger a_2^\dagger\tanh\lambda]\exp[a_3^\dagger a_4^\dagger\tanh\mu]|0000\rangle.
\end{aligned} \tag{14}$$

Using the Baker–Hausdorff formula $e^A e^B=e^B e^A e^{[A,B]}$, valid for $[A,[A,B]]=[B,[A,B]]=0$, we have

$$\begin{aligned}
&\exp[-a_1^\dagger a_4\cosh\lambda\sinh\mu]\exp[a_3^\dagger a_4^\dagger\tanh\mu]\\
&\quad=\exp[a_3^\dagger a_4^\dagger\tanh\mu]\exp[-a_1^\dagger a_3^\dagger\sinh\mu\tanh\mu\cosh\lambda]\\
&\qquad\times\exp[-a_1^\dagger a_4\cosh\lambda\sinh\mu],
\end{aligned} \tag{15}$$

and substituting (15) into (14) we obtain

$$\begin{aligned}
&S_{12}S_{34}\exp[a_1^\dagger a_3^\dagger]|0000\rangle\\
&\quad=\operatorname{sech}\lambda\operatorname{sech}\mu\exp[a_1^\dagger a_2^\dagger\tanh\lambda+a_3^\dagger a_4^\dagger\tanh\mu+a_1^\dagger a_3^\dagger\operatorname{sech}\lambda\operatorname{sech}\mu]|0000\rangle\\
&\quad\equiv|\rangle_{\lambda,\mu}.
\end{aligned} \tag{16}$$

Physically, the transformation (12–16) corresponds to the following nonlinear optical process: the $a_1^\dagger$-mode (signal mode) and $a_2^\dagger$-mode (idler mode) get coupled in a PDC where squeezing happens in a frequency domain). The case is true for $a_3^\dagger$ and $a_4^\dagger$ modes, too. Thus $|\rangle_{\lambda,\mu}$ can be implemented by combining a beam splitter and two parametric amplifiers. Then making a three-mode displacement $D_1(\alpha)D_2(\beta)D_4(\gamma)$ for $|\rangle_{\lambda,\mu}$, where $D_1(\alpha)=\exp[\alpha a_1^\dagger-\alpha^* a_1]$, we make up the new entangled state

$$D_1(\alpha)D_2(\beta)D_4(\gamma)|\rangle_{\lambda,\mu}=|\alpha,\beta,\gamma\rangle_{\lambda,\mu}. \tag{17}$$

One can achieve the displacement $D_1(\alpha)$ experimentally by reflecting the light field from an almost perfect reflecting mirror and adding through the mirror a field (a strong coherent local oscillator) that has been phase- and amplitude-modulated according to the value α, and a similar analysis is true for $D_2(\beta)$ and $D_4(\gamma)$.

4 Properties of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$

We now investigate the major properties of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$. Using the normally ordered form of the vacuum projection operator

$$|0000\rangle\langle 0000| = :\exp[-a_1^\dagger a_1 - a_2^\dagger a_2 - a_3^\dagger a_3 - a_4^\dagger a_4]:, \tag{18}$$

where the symbol : : denotes normal ordering, and the following integral formula

$$\int \frac{d^2 z}{\pi}\exp[\xi|z|^2+\zeta z+\eta z^*] = -\frac{1}{\xi}\exp\left[-\frac{\zeta\eta}{\xi}\right], \quad \mathrm{Re}\,\xi<0, \tag{19}$$

we can employ the IWOP technique [26–31] to prove the completeness relation of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$

$$\begin{aligned}
&\cosh^2\mu \int \frac{d^2\alpha d^2\beta d^2\gamma}{\pi^3}|\alpha,\beta,\gamma\rangle_{\lambda,\mu\lambda,\mu}\langle\alpha,\beta,\gamma| \\
&\quad = :\exp\left[\frac{1}{\mathrm{sech}^2\lambda}(a_2^\dagger - a_1\tanh\lambda + A\tanh\lambda)(a_2 - a_1^\dagger\tanh\lambda + A^\dagger\tanh\lambda)\right] \\
&\qquad \times \exp[(a_4^\dagger - a_3\tanh\mu)(a_4 - a_3^\dagger\tanh\mu) + A^\dagger A + A_1 + A_2 + A_3 + K]: \\
&\quad = :\exp[0]: = 1,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
&K = -a_1^\dagger a_1 - a_2^\dagger a_2 - a_3^\dagger a_3 - a_4^\dagger a_4, \\
&A_1 = (a_1^\dagger a_2^\dagger + a_1 a_2)\tanh\lambda, \qquad A^\dagger = a_1^\dagger - a_2\tanh\lambda - a_3\,\mathrm{sech}\,\lambda\,\mathrm{sech}\,\mu, \\
&A_2 = (a_3^\dagger a_4^\dagger + a_3 a_4)\tanh\mu, \qquad A_3 = (a_1^\dagger a_3^\dagger + a_1 a_3)\,\mathrm{sech}\,\lambda\,\mathrm{sech}\,\mu.
\end{aligned} \tag{21}$$

so the factor $\cosh^2\mu$ is integration measurement needed for the completeness relation. It is worthy of pointing out that here the integration is in three-fold, not in four-fold, this is because the state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ is entangled among four modes, which reduces the folds of integration. Thus we notice that although $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ is defined in four-mode Fock space, due to its four modes are mutually entangled, it spans a completeness relation with three-fold complex integral measurement. To derive the exact expression of ${}_{\lambda,\mu}\langle\alpha',\beta',\gamma'|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$, we use the overcompleteness relation of four-mode coherent state [32–38]

$$\int\prod_{i=1}^{4}\frac{d^2 z_i}{\pi}|z_1,z_2,z_3,z_4\rangle\langle z_1,z_2,z_3,z_4| = 1, \tag{22}$$

where

$$|z_1, z_2, z_3, z_4\rangle = \exp\left[\sum_{i=1}^{4}\left(-\frac{1}{2}|z_i|^2 + z_i a_i^{\dagger}\right)\right]|0000\rangle, \tag{23}$$

and

$$\begin{aligned}
&\langle z_1, z_2, z_3, z_4|\alpha, \beta, \gamma\rangle_{\lambda,\mu} \\
&\quad = \operatorname{sech}\lambda \operatorname{sech}\mu \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{4}|z_i|^2 + |\alpha|^2 + |\beta|^2 + |\gamma|^2\right)\right. \\
&\qquad \left. + (z_1^* - \alpha^*)(z_2^* - \beta^*)\tanh\lambda\right] \\
&\qquad \times \exp[z_3^*(z_4^* - \gamma^*)\tanh\mu + (z_1^* - \alpha^*)z_3^* \operatorname{sech}\lambda \operatorname{sech}\mu + \alpha z_1^* + \beta z_2^* + \gamma z_4^*],
\end{aligned} \tag{24}$$

to calculate

$$\begin{aligned}
&{}_{\lambda',\mu'}\langle \alpha', \beta', \gamma'|\alpha, \beta, \gamma\rangle_{\lambda,\mu} \\
&\quad = \int \prod_{i=1}^{4}\left(\frac{d^2 z_i}{\pi}\right)_{\lambda',\mu'}\langle \alpha', \beta', \gamma'|z_1, z_2, z_3, z_4\rangle\langle z_1, z_2, z_3, z_4|\alpha, \beta, \gamma\rangle_{\lambda,\mu} \\
&\quad = \frac{B \operatorname{sech}\lambda \operatorname{sech}\mu \operatorname{sech}\lambda' \operatorname{sech}\mu'}{F - (1 - \tanh\lambda\tanh\lambda')\tanh\mu\tanh\mu'} \exp\left[\frac{1}{1 - \tanh\lambda\tanh\lambda'}C_1 C_2\right] \\
&\qquad \times \exp\left[\frac{1 - \tanh\lambda\tanh\lambda'}{F}C_5 C_6 + \frac{F C_3 C_4}{F - (1 - \tanh\lambda\tanh\lambda')\tanh\mu\tanh\mu'}\right] \\
&\qquad \times \exp[(\alpha - \beta^*\tanh\lambda)(\alpha'^* - \beta'\tanh\lambda') + \alpha^*\beta^*\tanh\lambda + \alpha'\beta'\tanh\lambda'],
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
C_1 &= \beta'^* + (\alpha - \alpha' - \beta^*\tanh\lambda)\tanh\lambda', \\
C_2 &= \beta + (\alpha'^* - \alpha^* - \beta'\tanh\lambda')\tanh\lambda, \\
C_3 &= \gamma'^* + \frac{1 - \tanh\lambda\tanh\lambda'}{F}C_5\tanh\mu', \\
C_4 &= \gamma + \frac{1 - \tanh\lambda\tanh\lambda'}{F}C_6\tanh\mu, \\
C_5 &= \frac{\alpha'^* - \alpha^* - (\beta' - \beta)\tanh\lambda'}{1 - \tanh\lambda\tanh\lambda'}\operatorname{sech}\lambda \operatorname{sech}\mu - \gamma^*\tanh\mu, \\
C_6 &= \frac{\alpha - \alpha' - (\beta^* - \beta'^*)\tanh\lambda}{1 - \tanh\lambda\tanh\lambda'}\operatorname{sech}\lambda' \operatorname{sech}\mu' - \gamma'\tanh\mu', \\
F &= 1 - \tanh\lambda\tanh\lambda' - \operatorname{sech}\lambda \operatorname{sech}\mu \operatorname{sech}\lambda' \operatorname{sech}\mu', \\
B &= \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\alpha'|^2 + |\beta'|^2 + |\gamma'|^2)\right].
\end{aligned} \tag{26}$$

Equation (25) shows that $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ state vectors are not orthogonal to each other. For simplicity, we further assume that the two PDCs are identical ($\lambda = \mu = \lambda' = \mu'$), then the orthogonal relation in (25) is reduced to

$$
\begin{aligned}
&{}_{\lambda,\lambda}\langle\alpha',\beta',\gamma'|\alpha,\beta,\gamma\rangle_{\lambda,\lambda} \\
&\quad = BC\operatorname{sech}^4\lambda \lim_{\epsilon\to 0}\frac{1}{\epsilon}\exp\left[-\frac{\operatorname{sech}^2\lambda}{\epsilon}|\alpha'-\alpha-(\beta'^*-\beta^*)\tanh\lambda\right. \\
&\qquad \left. +(\gamma'-\gamma)\tanh\lambda|^2\right]
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
C = {} & \exp[\cosh^2\lambda(\beta'^*+(\alpha-\alpha')\tanh\lambda-\beta^*\tanh^2\lambda) \\
& \times(\beta+(\alpha'^*-\alpha^*)\tanh\lambda-\beta'\tanh^2\lambda)] \\
& \times\exp[\coth^2\lambda(\alpha-\alpha'-(\beta^*-\beta'^*+\gamma')\tanh\lambda) \\
& \times(\alpha'^*-\alpha^*-(\beta'-\beta+\gamma^*)\tanh\lambda)] \\
& \times\exp[(\alpha-\beta^*\tanh\lambda)(\alpha'^*-\beta'\tanh\lambda')+(\alpha^*\beta^*+\alpha'\beta')\tanh\lambda].
\end{aligned}
\tag{28}
$$

Using the limiting formula of δ-function

$$
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\exp\left[-\frac{1}{\epsilon}|z|^2\right]=\pi\delta(z)\delta(z^*)=\pi\delta(x)\delta(y),\qquad z=x+\mathrm{i}y,
\tag{29}
$$

we obtain

$$
\begin{aligned}
&{}_{\lambda,\lambda}\langle\alpha',\beta',\gamma'|\alpha,\beta,\gamma\rangle_{\lambda,\lambda} \\
&\quad =\pi BC\operatorname{sech}^2\lambda\,\delta(\alpha'-\alpha-(\beta'^*-\beta^*)\tanh\lambda+(\gamma'-\gamma)\tanh\lambda) \\
&\qquad \times\delta(\alpha'^*-\alpha^*-(\beta'-\beta)\tanh\lambda+(\gamma'^*-\gamma^*)\tanh\lambda).
\end{aligned}
\tag{30}
$$

From (30) we see that $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ is partly orthogonal.

5 Quadrature Amplitude Measurement on $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$

We can more explicitly show that $|\alpha,\beta,\gamma\rangle_{\lambda\mu}$ is an entangled state by making the following three-fold Fourier transform,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{d}\alpha_2\mathrm{d}\beta_2\mathrm{d}\gamma_2 e^{\mathrm{i}(u\alpha_2+v\beta_2+\sigma\gamma_2)}|\alpha,\beta,\gamma\rangle_{\lambda,\mu} \\
&\quad = W(u,v,\sigma)|x_1\rangle_1\otimes|x_2\rangle_2\otimes|x_3\rangle_3\otimes|x_4\rangle_4,
\end{aligned}
\tag{31}
$$

where $|x_i\rangle_i$ all belong to the set of coordinate eigenvectors

$$
|x_i\rangle_i=\pi^{-1/4}\exp\left[-\frac{x_i^2}{2}+\sqrt{2}x_i a_i^\dagger-\frac{1}{2}a_i^{\dagger 2}\right]|0\rangle_i,
\tag{32}
$$

and $W(u,v,\sigma)$ is of less important factor, so we need not to write down it explicitly,

$$
\begin{aligned}
&x_1=(\alpha_1-u)/\sqrt{2},\qquad x_2=(\beta_1-v)/\sqrt{2},\qquad x_4=(\gamma_1-\sigma)/\sqrt{2}, \\
&x_3=\frac{1}{\sqrt{2}}[((\beta_1+v)\sinh\lambda-(\alpha_1+u)\cosh\lambda)\operatorname{sech}\mu-(\gamma_1+\sigma)\tanh\mu].
\end{aligned}
\tag{33}
$$

Then making the inverse Fourier transform of (31) we obtain

$$|\alpha,\beta,\gamma\rangle_{\lambda,\mu}=\int_{-\infty}^{\infty}\frac{\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma}{8\pi^3}e^{-\mathrm{i}(u\alpha_2+v\beta_2+\sigma\gamma_2)}\times W(u,v,\sigma)|x_1\rangle_1\otimes|x_2\rangle_2\otimes|x_3\rangle_3\otimes|x_4\rangle_4 \tag{34}$$

which confirms that $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ itself is an entangled state according to the standard theory of Schmidt decomposition [39]. When making a single-mode quadrature amplitude measurement, say, $|x\rangle_{11}\langle x|$, on $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$, where $|x\rangle$ is the coordinate eigenstate of X operator, using (34) we obtain

$$\begin{aligned}&{}_1\langle x|\alpha,\beta,\gamma\rangle_{\lambda,\mu}\\&=\sqrt{2}e^{-\mathrm{i}u\alpha_2}\int_{-\infty}^{\infty}\frac{\mathrm{d}v\mathrm{d}\sigma}{8\pi^3}e^{-\mathrm{i}(v\beta_2+\sigma\gamma_2)}W(v,\sigma)|x_2\rangle_2\otimes|x_3\rangle_3\otimes|x_4\rangle_4|_{u=\alpha_1-\sqrt{2}x},\end{aligned} \tag{35}$$

which shows that the remaining three modes are kept in entanglement. If a joint Bell measurement, say $|\eta'\rangle_{1,21,2}\langle\eta'|$, is performed on $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$, the result of measurement is

$${}_{1,2}\langle\eta'|\alpha,\beta,\gamma\rangle_{\lambda,\mu}=e^M\frac{\operatorname{sech}\lambda\operatorname{sech}\mu}{1-\tanh\lambda}\exp[\tau a_3^\dagger+\gamma a_4^\dagger+a_3^\dagger a_4^\dagger\tanh\mu]|00\rangle, \tag{36}$$

where e^M is an unimportant coefficient and

$$\tau=\frac{\operatorname{sech}\lambda\operatorname{sech}\mu}{1-\tanh\lambda}(\eta'^*-\alpha^*+\beta)-\gamma^*\tanh\mu. \tag{37}$$

Equation (36) indicates that after the Bell joint measurement the remaining two-mode is still in entanglement.

6 Application of the Entangled State $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$

Following Bennett et al. [2] we consider an application of the $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ in quantum teleportation. From (20) we know that any 4-mode state $|\ \rangle_{1234}$ can be expanded as

$$|\ \rangle_{1234}=\int\frac{\mathrm{d}^2\alpha\mathrm{d}^2\beta\mathrm{d}^2\gamma}{\pi^3}G(\alpha,\beta,\gamma,\lambda,\mu)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}, \tag{38}$$

where $G(\alpha,\beta,\gamma,\lambda,\mu)$ is the expansion coefficient

$$G(\alpha,\beta,\gamma,\lambda,\mu)=\cosh^2\mu\ {}_{\lambda,\mu}\langle\alpha,\beta,\gamma|\ \rangle_{1234}. \tag{39}$$

If Alice is able to teleport the 4-mode state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ to Bob, then she can teleport $|\ \rangle_{1234}$ since it can be expanded by the set of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$. Assuming Alice and Bob share a quantum channel composing of four bipartite entangled states [40], $|\eta_a\rangle_{56}\otimes|\eta_b\rangle_{78}\otimes|\eta_c\rangle_{9,10}\otimes|\eta_d\rangle_{11,12}$, which means that Alice owns particles 5, 7, and 9, 11, while Bob owns 6, 8, and 10, 12. Then the initial state of the whole system is $|\Psi_{in}\rangle=|\alpha,\beta,\gamma\rangle_{\lambda,\mu1234}\otimes|\eta_a\rangle_{56}\otimes|\eta_b\rangle_{78}\otimes|\eta_c\rangle_{9,10}\otimes|\eta_d\rangle_{11,12}$. Alice carries out a joint Bell measurement, the projection state is $|\eta\rangle_{15}\otimes|\eta'\rangle_{27}\otimes|\eta''\rangle_{39}\otimes|\eta'''\rangle_{4,11}$, then using the Schmidt decomposition of $|\eta\rangle_{15}$,

$${}_{15}\langle\eta|=e^{\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}x_1\langle x|\otimes\ {}_5\langle x-\eta_1|.e^{-\mathrm{i}\eta_2 x}, \tag{40}$$

and using (34) we can calculate

$$\begin{aligned}|\Psi_{\rm out}\rangle &\equiv {}_{4,11}\langle\eta'''|\otimes\ {}_{39}\langle\eta''|\otimes_{27}\langle\eta'|\otimes_{15}\langle\eta|\Psi_{in}\rangle\\ &= e^{\mathrm{i}\Phi}\int_{-\infty}^{\infty}\frac{\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma}{8\pi^3}e^{-\mathrm{i}(u\alpha_2+v\beta_2+\sigma\gamma_2)}W(u,v,\sigma)\\ &\quad\times e^{\mathrm{i}((\eta_{a2}-\eta_2)x_1-\eta_{a2}\eta_1)}|x_1-\eta_1-\eta_{a1}\rangle_6\\ &\quad\otimes e^{\mathrm{i}((\eta_{b2}-\eta_2')x_2-\eta_{b2}\eta_1')}|x_2-\eta_1'-\eta_{b1}\rangle_8\\ &\quad\otimes e^{\mathrm{i}((\eta_{c2}-\eta_2'')x_3-\eta_{c2}\eta_1'')}|x_3-\eta_1''-\eta_{c1}\rangle_{10}\\ &\quad\otimes e^{\mathrm{i}((\eta_{d2}-\eta_2''')x_4-\eta_{d2}\eta_1''')}|x_4-\eta_1'''-\eta_{d1}\rangle_{12},\end{aligned}\tag{41}$$

where $e^{\mathrm{i}\Phi}=e^{\mathrm{i}(\eta_1\eta_2+\eta_1'\eta_2'+\eta_1''\eta_2''+\eta_1'''\eta_2'''-\eta_{a1}\eta_{a2}-\eta_{b1}\eta_{b2}-\eta_{c1}\eta_{c2}-\eta_{d1}\eta_{d2})/2}$. Note that

$$\begin{aligned}&e^{\mathrm{i}((\eta_{a2}-\eta_2)x_1-\eta_{a2}\eta_1)}|x_1-\eta_1-\eta_{a1}\rangle_6\\ &\quad= e^{\mathrm{i}((\eta_{a2}-\eta_2)X_6+\eta_{a2}\eta_{a1}-\eta_2(\eta_1+\eta_{a1}))}e^{\mathrm{i}P_6(\eta_1+\eta_{a1})}|x_1\rangle_6,\end{aligned}\tag{42}$$

$$\begin{aligned}&e^{\mathrm{i}((\eta_{b2}-\eta_2')x_2-\eta_{b2}\eta_1')}|x_2-\eta_1'-\eta_{b1}\rangle_8\\ &\quad= e^{\mathrm{i}((\eta_{b2}-\eta_2')X_8+\eta_{b1}\eta_{b2}-\eta_2'(\eta_1'+\eta_{b1}))}e^{\mathrm{i}P_8(\eta_1'+\eta_{b1})}|x_2\rangle_8,\end{aligned}\tag{43}$$

$$\begin{aligned}&e^{\mathrm{i}((\eta_{c2}-\eta_2'')x_3-\eta_{c2}\eta_1'')}|x_3-\eta_1''-\eta_{c1}\rangle_{10}\\ &\quad= e^{\mathrm{i}((\eta_{c2}-\eta_2'')X_{10}+\eta_{c1}\eta_{c2}-\eta_2''(\eta_1''+\eta_{c1}))}e^{\mathrm{i}P_{10}(\eta_1''+\eta_{c1})}|x_3\rangle_{10},\end{aligned}\tag{44}$$

$$\begin{aligned}&e^{\mathrm{i}((\eta_{d2}-\eta_2''')x_4-\eta_{d2}\eta_1''')}|x_4-\eta_1'''-\eta_{d1}\rangle_{12}\\ &\quad= e^{\mathrm{i}((\eta_{d2}-\eta_2''')X_{12}+\eta_{d1}\eta_{d2}-\eta_2'''(\eta_1'''+\eta_{d1}))}e^{\mathrm{i}P_{12}(\eta_1'''+\eta_{d1})}|x_4\rangle_{12}.\end{aligned}\tag{45}$$

Substituting (42–45) into (41) we obtain

$$\begin{aligned}|\Psi_{out}\rangle &= e^{\mathrm{i}\Phi'}U_xU_p\int_{-\infty}^{\infty}\frac{\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma}{8\pi^3}e^{-\mathrm{i}(u\alpha_2+v\beta_2+\sigma\gamma_2)}W(u,v,\sigma)|x_1\rangle_6\\ &\qquad\otimes|x_2\rangle_8\otimes|x_3\rangle_{10}\otimes|x_4\rangle_{12}\\ &= e^{\mathrm{i}\Phi'}U_xU_p|\alpha,\beta,\gamma\rangle_{\lambda,\mu 6,8,10,12},\end{aligned}\tag{46}$$

where

$$\begin{aligned}e^{\mathrm{i}\Phi'} &= e^{\mathrm{i}(\eta_{a1}\eta_{a2}+\eta_{b1}\eta_{b2}+\eta_{c1}\eta_{c2}+\eta_{d1}\eta_{d2}-\eta_1\eta_2-\eta_1'\eta_2'-\eta_1''\eta_2''-\eta_1'''\eta_2''')/2}\\ &\quad\times e^{-\mathrm{i}(\eta_2\eta_{a1}+\eta_2'\eta_{b1}+\eta_2''\eta_{c1}+\eta_2'''\eta_{d1})},\end{aligned}\tag{47}$$

$$U_x = e^{\mathrm{i}(\eta_{a2}-\eta_2)X_6}e^{\mathrm{i}(\eta_{b2}-\eta_2')X_8}e^{\mathrm{i}(\eta_{c2}-\eta_2'')X_{10}}e^{\mathrm{i}(\eta_{d2}-\eta_2''')X_{12}},\tag{48}$$

$$U_p = e^{\mathrm{i}P_6(\eta_1+\eta_{a1})}e^{\mathrm{i}P_8(\eta_1'+\eta_{b1})}e^{\mathrm{i}P_{10}(\eta_1''+\eta_{c1})}e^{\mathrm{i}P_{12}(\eta_1'''+\eta_{d1})}.\tag{49}$$

Then Alice informs Bob of the data of η, η', η'' and η''' via a classical channel, and Bob, up to an over phase factor, makes the unitary transform $(U_xU_p)^{-1}$, and successfully obtains the unknown state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$.

7 Conclusion

In summary, we have introduced the four-mode entangled state $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ of continuous variables both theoretically and experimentally and analyzed its Schmidt decomposition as well as a quadrature amplitude measurement on it. We have proved the completeness relation of $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ and showed that $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ is partly orthogonal. A feasible experimental setup—a symmetrical BS and a pair of PDCs is proposed to generate such a four-mode entangled state. Its application in quantum information is briefly described.

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